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Hamiltonian and Path Integral Quantization of the Conformally Gauge-Fixed Polyakov D1 Brane Action in the Presence of a Scalar Dilation Field

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Abstract The conformally gauge-fixed Polyakov D1 brane action in the presence of a scalar dilaton field is seen to be a constrained system in the sense of Dirac. In the present work we study its Hamiltonian and path integral quantization in the instant-form of dynamics using the equal world-sheet time framework.

Keywords Dirac quantization · Hamiltonian quantization · Path integral quantization · D-brane actions · Polyakov action

1 Introduction

The Polyakov action is almost the starting point in any studies on string theories [1–12] and it is therefore also one of the most widely studied and discussed topics in this field in the recent times [1–19]. The conformally gauge-fixed Polyakov D1 brane action without a scalar dilaton field is seen to be an unconstrained system in the sense of Dirac [9–19]. However, in the presence of a scalar dilaton field it is seen to be a constrained system in the sense of Dirac [9–19], possessing one primary and one secondary Gauss law constraint [9–19]. In the present work we study the Hamiltonian and path integral quantization of this theory in the conformal gauge (CG), in the presence of a scalar dilation field in the usual instant-form (IF) of dynamics, using the equal world-sheet (WS)-time (EWST) framework, on the

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hyperplanes defined by the WS-time $\sigma^0 = \tau =$ constant [9–23] using the standard constraint quantization techniques [9–23]. In the next section we briefly recapitulate some basics of the model without the scalar dilation field in the CG. In Sect. 3, we study the Hamiltonian and path integral formulations of this conformally gauge-fixed model in the presence of the scalar dilation field for the D1 brane action. The summary and discussion is finally given in Sect. 4.

2 Action without a Scalar Dilaton Field

The Polyakov action describing the propagation of a D1 brane in a d-dimensional curved background $h_{\alpha\beta}$ (with $d = 10$ for the fermionic and $d = 26$ for bosonic D1 brane) is defined by [1–12]:

$$\tilde{S} = \int \tilde{\mathcal{L}} d^2\sigma \tag{1a}$$

$$\tilde{\mathcal{L}} = \left[-\frac{T}{2}\sqrt{-h}h^{\alpha\beta}G_{\alpha\beta}\right]; \quad h = \det(h_{\alpha\beta}) \tag{1b}$$

$$G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}; \quad \eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1) \tag{1c}$$

$$\mu, \nu = 0, 1, \ldots, (d-1); \quad \alpha, \beta = 0, 1 \tag{1d}$$

Here $\sigma^\alpha \equiv (\tau, \sigma)$ are the two parameters describing the world-sheet (WS). The overdots and primes would denote the derivatives with respect to τ and σ. T is the string tension. $G_{\alpha\beta}$ is the induced metric on the WS and $X^\mu(\tau, \sigma)$ are the maps of the WS into the d-dimensional Minkowski space and describe the strings evolution in space-time [1–12]. $h_{\alpha\beta}$ are the auxiliary fields (which turn out to be proportional to the metric tensor $\eta_{\alpha\beta}$ of the two-dimensional surface swept out by the string). One can think of $\tilde{S}$ as the action describing d massless scalar fields X^μ in two dimensions moving on a curved background $h_{\alpha\beta}$. Also because the metric components $h_{\alpha\beta}$ are varied in (1), the 2-dimensional gravitational field $h_{\alpha\beta}$ is treated not as a given background field, but rather as an adjustable quantity coupled to the scalar fields [1–8]. The action $\tilde{S}$ has the well-known three local gauge symmetries given by the 2-dimensional WS reparametrization invariance (WSRI) and the Weyl invariance (WI) [1–12]. We could now use the three local gauge symmetries of the theory to choose $h_{\alpha\beta}$ to be of a particular form [1–12]:

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \tag{2}$$

This is the so-called conformal gauge (CG). In this CG we have

$$\sqrt{-h} = \sqrt{-\det(h_{\alpha\beta})} = +1 \tag{3}$$

and the action $\tilde{S}$ in this CG now becomes:

$$S_1 = \int \mathcal{L}_1 d^2\sigma \tag{4a}$$

$$\begin{aligned} \mathcal{L}_1 &= (-T/2)\sqrt{-h}h^{\alpha\beta}G_{\alpha\beta} \\ &= (-T/2)\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \end{aligned} \tag{4b}$$

$$
\begin{aligned}
&= (-T/2)\partial^\beta X^\mu \partial_\beta X_\mu \\
&= (-T/2)\left[\partial_\sigma X^\mu \partial_\sigma X_\mu - \partial_\tau X^\mu \partial_\tau X_\mu\right] \\
&= (-T/2)\left[(X')^2 - (\dot{X})^2\right]
\end{aligned}
$$

$$
\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau} \quad \text{and} \quad X'^\mu = \frac{\partial X^\mu}{\partial \sigma} \tag{4c}
$$

Now onwards in this section, we would study the D1 brane described by the action (4). The canonical momenta conjugate to X^μ obtained from (4) are:

$$
P^\mu := \frac{\partial \mathcal{L}_1}{\partial(\partial_\tau X^\mu)} = T\partial_\tau X^\mu \tag{5}
$$

Here the velocities $\partial_\tau X^\mu = P^\mu/T$ are expressible. The canonical Hamiltonian density corresponding to $\mathcal{L}_1$ is

$$
\mathcal{H}_1^c = \left[P^\mu(\partial_\tau X_\mu) - \mathcal{L}_1\right] = \left[P^2/(2T) + \frac{T}{2}(X')^2\right] \tag{6}
$$

The quantization of the system is trivial. The nonvanishing equal WS-time (EWST) commutation relations for the theory are obtained as [1–12]:

$$
[X^\mu(\sigma,\tau),\ P_\nu(\sigma',\tau)] = i\delta^\mu_\nu\ \delta(\sigma-\sigma') \tag{7}
$$

where $\delta(\sigma-\sigma')$ is the one-dimensional Dirac distribution function.

Further, in the path integral formulation, the transition to the quantum theory is made by writing the vacuum to vacuum transition amplitude for the theory called the generating functional $Z_1[J\mu]$ of the theory in the presence of external sources J_μ as [1–19]:

$$
Z_1[J_\mu] := \int [d\mu] \exp\left[i\int d^2\sigma[\mathcal{L}_1^{IO} + J_\mu X^\mu]\right] \tag{8}
$$

where the functional measure $[d\mu]$ is defined by [9–19]:

$$
[d\mu] = [dX^\mu][dP_\mu] \tag{9}
$$

and the first oder Lagrangian density is defined by:

$$
\begin{aligned}
\mathcal{L}_1^{IO} &= \left[P^\mu(\partial_\tau X_\mu) - \mathcal{H}_1^c\right] \\
&= \left[\frac{1}{2T}P^2 - \frac{T}{2}(X')^2\right]
\end{aligned} \tag{10}
$$

It is obvious from the above considerations that the above theory is unconstrained in the sense of Dirac [9–19]. However, when this theory is considered in the presence of a scalar dilaton field as we do in the next section then it becomes a constrained system in the sense of Dirac [9–19]. Now we study this theory in the presence of a scalar dilation field in the following section.

3 Action in the Presence of a Scalar Dilation Field

The Polyakov D1 brane action in the CG describing the propagation of a D-string in a d-dimensional flat background $h_{\alpha\beta}$ (given by (2)) in the presence of a scalar dilation field $\varphi(\equiv \varphi(\sigma,\tau))$ is defined by [1–12]:

$$S_2 = \int \mathcal{L}_2 d^2\sigma \tag{11a}$$

$$\mathcal{L}_2 = [e^{-\varphi}\mathcal{L}_1] \tag{11b}$$

$$= \left[-\frac{1}{2}Te^{-\varphi}\sqrt{-h}h^{\alpha\beta}G_{\alpha\beta}\right]$$

$$= \left[-\frac{1}{2}Te^{-\varphi}\right]\left[\partial^{\beta}X^{\mu}\partial_{\beta}X_{\mu}\right] \tag{11c}$$

$$= \left[-\frac{1}{2}Te^{-\varphi}\right]\left[(X')^2-(\dot{X})^2\right] \tag{11d}$$

The canonical momenta conjugate to X^{μ} obtained from $\mathcal{L}_2$ are:

$$P^{\mu} := \frac{\partial\mathcal{L}_2}{\partial(\partial_{\tau}X^{\mu})} = Te^{-\varphi}\partial_{\tau}X^{\mu} \tag{12a}$$

$$\pi = \frac{\partial\mathcal{L}_2}{\partial(\partial_{\tau}\varphi)} = 0 \tag{12b}$$

where P^{μ} and π are the momenta conjugate respectively to X_{μ} and φ. The above definitions of momenta lead to the existence of one primary constraint [9–19] given by

$$\rho_1 = \pi \approx 0 \tag{13}$$

Here the symbol $\approx$ denotes a weak equality (WE) in the sense of Dirac [9–19], and it implies that these above constraints hold as strong equalities only on the reduced hypersurface of the constraints and not in the rest of the phase space of the classical theory (and similarly one can consider it as a weak operator equality (WOE) for the corresponding quantum theory) [9–23].

The canonical Hamiltonian density corresponding to $\mathcal{L}_2$ is

$$\mathcal{H}_2^c = \left[P^{\mu}\dot{X}_{\mu} + \pi\dot{\varphi} - \mathcal{L}_2\right]$$

$$= \left[e^{\varphi}P^2/(2T) + \frac{1}{2}Te^{-\varphi}(X')^2\right] \tag{14}$$

After incorporating the primary constraint of the theory ρ_1 into the canonical Hamiltonian density of the system with the help of a Lagrange multiplier $u(\tau,\sigma)$, the total Hamiltonian density of the system could be written as

$$\mathcal{H}_2^T = \left[e^{\varphi}P^2/(2T) + \frac{1}{2}Te^{-\varphi}(X')^2 + u\rho_1\right] \tag{15}$$

we will now treat $u(\tau,\sigma)$ as dynamical. The Hamiltonian equation obtained from the total Hamiltonian

$$H_2^T = \int \mathcal{H}_2^T d\sigma \tag{16}$$

e.g., for the closed strings with periodic boundary conditions (BC's) are:

$$\begin{aligned}
\partial_\tau X^\mu &= \frac{\partial H_2^T}{\partial P_\mu} = e^\varphi P/T \\
-\partial_\tau P^\mu &= \frac{\partial H_2^T}{\partial X_\mu} = (-T)[e^{-\varphi} X''^\mu - e^{-\varphi}\varphi' X'^\mu] \\
\partial_\tau \varphi &= \frac{\partial H_2^T}{\partial \pi} = u \\
-\partial_\tau \pi &= \frac{\partial H_2^T}{\partial \varphi} = \left[e^\varphi P^2/(2T) - \frac{1}{2} T e^{-\varphi}(X')^2\right] \\
\partial_\tau u &= \frac{\partial H_2^T}{\partial p_u} = 0 \\
-\partial_\tau p_u &= \frac{\partial H_2^T}{\partial u} = \pi
\end{aligned} \tag{17}$$

These are the equation of motion of the theory that preserve the constraints of the theory in the course of time. Demanding that the primary constraint ρ_1 be preserved in the course of time one obtains the secondary Gauss law constraint:

$$\rho_2 = \{\rho_1, \mathcal{H}_2^T\}_p = \left[-e^\varphi P^2/(2T) + \frac{1}{2} T e^{-\varphi}(X')^2\right] \approx 0 \tag{18}$$

Now the preservation of ρ_2 for all time does not lead to any further constraint. The existence of this constraint in the present theory is analogous to the existence of the secondary Gauss law constraint in the usual free electrodynamics. It may be important to emphasize here that this secondary constraint given by the above equation (18), has to be solved for the dilaton field in terms of the string fields (X^μ) and their canonically conjugate momenta (P_μ). As a consequence of this, the dilaton field and its canonically conjugate momentum (π) get eliminated when one considers the Dirac brackets of the theory, as would be shown later where it would become evident that only the string fields (X^μ) and their canonically conjugate momenta (P_μ) eventually survive in the reduced phase space of the theory.

The system is thus seen to possess only two constraints ρ_1 and ρ_2. The matrix of the Poisson brackets of the constraints ρ_1 and ρ_2, namely, $R_{\alpha\beta}(\sigma, \sigma') := \{\rho_\alpha(\sigma), \rho_\beta(\sigma')\}_p$ is now calculated and its nonvanishing elements are obtained as

$$R_{12} = -R_{21} = [1/(2T)] R e^\varphi \delta(\sigma - \sigma') \tag{19a}$$

$$R_{22} = 2(X' \cdot P)\delta'(\sigma - \sigma') \tag{19b}$$

$$R = [P^2 + T^2 (X')^2 e^{-2\varphi}] \tag{19c}$$

The matrix $R_{\alpha\beta}$ is seen to be nonsingular and its determinant is obtained as:

$$\left[\| \det(R_{\alpha\beta}) \|\right]^{\frac{1}{2}} := [1/(2T)] R e^\varphi \delta(\sigma - \sigma') \tag{20}$$

The nonvanishing elements of the inverse of the matrix $R_{\alpha\beta}$ (i.e., the elements of the matrix $(R^{-1})_{\alpha\beta}$) are obtained as:

$$(R^{-1})_{11} = [8/R^2] T^2 e^{-2\varphi} (X' \cdot P)\delta'(\sigma - \sigma') \tag{21a}$$

$$(R^{-1})_{12} = -(R^{-1})_{21} = [-2T/R] e^{-\varphi} \delta(\sigma - \sigma') \tag{21b}$$

with

$$\int R(\sigma,\sigma'')R^{-1}(\sigma'',\sigma')d\sigma'' = \mathbf{1}_{2\times 2}\delta(\sigma-\sigma') \tag{22}$$

The nonsingular nature of the matrix $R_{\alpha\beta}$ signifies that the set of constraints ρ_i is second class [24] and consequently the theory described by the action S_2 is gauge noninvariant (GNI) [9–23] as is expected because it is gauge-fixed theory (under the conformal gauge given by (2)) and therefore GNI.

It is important to recapitulate here that the original Polyakov D1 brane action defined by the action $\tilde{S}$ described the propagation of the D-string in a curved background and is gauge-invariant (GI) possessing the well known three local (or gauge) symmetries defined by the 2-dimensional WS reparametrization invariance and the Weyl invariance [1–12]. However, the present theory under our current investigation is GNI because it is gauge-fixed theory being studied under the so-called conformal gauge defined by (2) and is consequently GNI as it should be.

Now, following the Dirac quantization procedure in the Hamiltonian formulation [9–19] the nonvanishing equal WS-time Dirac brackets of the theory are formally obtained as:

$$\{X^\mu(\sigma,\tau), P_\nu(\sigma',\tau)\}_{\rm DB} = \delta^\mu_\nu \delta(\sigma-\sigma') \tag{23a}$$

$$\{\varphi(\sigma,\tau), P^\mu(\sigma',\tau)\}_{\rm DB} = [2/R]T^2 e^{-2\varphi} X'^\mu \delta'(\sigma-\sigma') \tag{23b}$$

$$\{\varphi(\sigma,\tau), X^\mu(\sigma',\tau)\}_{\rm DB} = [2/R]P^\mu \delta(\sigma-\sigma') \tag{23c}$$

$$\{\varphi(\sigma,\tau), \varphi(\sigma',\tau)\}_{\rm DB} = [8/R^2]T^2 e^{-2\varphi}(X'\cdot P)\delta'(\sigma-\sigma') \tag{23d}$$

As discussed in the forgoing, it may be seen here that if one solves the secondary Gauss law constraint of the theory for the dilaton field in terms of the string fields (X^μ) and their canonically conjugate momenta (P_μ), then the dilaton field and its canonically conjugate momentum (π) get eliminated in the above Dirac brackets, and only the string fields (X^μ) and their canonically conjugate momenta (P_μ) eventually survive in the reduced phase space of the theory, as is clear from the above equation (cf. (23)).

Now in the canonical quantization of the theory while going from equal WS time (EWST) Dirac brackets of the theory to the corresponding EWST commutation relations one would encounter here the problem of operator ordering [24] because the product of canonical variable of the theory are involved in the classical description of the theory (like in the expressions for the constraints of the theory) as well as in the calculation of the Dirac brackets. These variables are envisaged as noncommuting operators in the quantized theory leading to the problem of so-called operator ordering [25]. This problem could, however, be resolved [24] by demanding that all the string fields and momenta of the theory are Hermitian operators and that all the canonical commutation relations be consistent with the Hermiticity of the operators [24].

In the path integral formulation, the transition to quantum theory is now made by writing the generating functional $Z_2[J_i]$ for the theory in the presence of the external sources J_i as [9–19]:

$$Z_2[J_i] := \int [d\mu]\exp\left[i\int d^2\sigma[\mathcal{L}_2^{\rm IO} + J_i\Phi^i]\right] \tag{24}$$

where the phase space variables of the theory defined by the action S_2 are $\Phi^i \equiv (X^\mu,\varphi)$ with the corresponding respectively canonical conjugate momenta: $\Pi_i \equiv (P_\mu,\pi)$. The functional measure $[d\mu]$ of the generating functional $Z_2[J_i]$ is obtained (using (13), (18) and (20)) as [9–19]:

$$[d\mu] = [1/(2T)][Re^{\varphi}\delta(\sigma - \sigma')][dx^{\mu}][d\varphi][dP_{\mu}][d\pi] \\ \times \delta[(\pi) \approx 0] \cdot \delta[[-e^{\varphi}P^2/(2T) + (T/2)e^{-\varphi}(X')^2] \approx 0] \tag{25}$$

and the first order Lagrangian density $\mathcal{L}_2^{\text{IO}}$ is defined by

$$\mathcal{L}_2^{\text{IO}} = [P^{\mu}(\partial_{\tau}X_{\mu}) + \pi(\partial_{\tau}\varphi) + p_u(\partial_{\tau}u) - \mathcal{H}_2^T] \\ = [[1/(2T)]e^{\varphi}P^2 - (T/2)e^{-\varphi}(X')^2] \tag{26}$$

It may be worthwhile to point out here that the evaluation of the above path integral given by (24) with the measure given by (25) is a non trivial task connected with the operator ordering problem which is a rather open problem and it needs to be studied further. The Hamiltonian and path integral quantization of the theory described by the action S_2 is now complete.

4 Summary and Discussion

The conformally gauge-fixed Polyakov D1 brane action without a scalar dilaton field is seen to be an unconstrained system in the sense of Dirac [9–19]. However, in the presence of a scalar dilaton field it is seen to be a constrained system in the sense of Dirac [9–19]. In this work we have studied the Hamiltonian and path integral quantization of this theory in the presence of the scalar dilation field using the standard constraint quantization techniques [9–23] in the usual instant-form of dynamics in the equal world-sheet time framework on the hyperplanes of the WS defined by: WS-time $\sigma^0 = \tau =$ constant [20–23] in the absence of boundary conditions [25]. The problem of operator ordering occurring here while making a transition from EWST Dirac brackets to the corresponding EWST commutation relations, can be resolved by demanding that all the string fields and momenta of the theory are Hermitian operators and that all the canonical commutation relations be consistent with the hermiticity of these operators [24] as explained earlier. It is important to mention here that in our work we have not imposed any boundary conditions for the open and closed strings separately. There are two ways to take these boundary conditions into account: (a) one way is to impose them directly in the usual way for the open and closed strings separately in an appropriate manner [1–12], (b) an alternative second way is to treat them as the Dirac primary constraints [25] and study the theory accordingly [25]. At present our related work is underway and would be reported later.

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